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PERTURBATION THEORY FOR DEGENERATE ACOUSTIC EIGENMODES

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Abstract - **A general formulation of the perturbation procedure applied to the eigenvalue problem for resonant frequencies in crystal resonators under biasing deformations is given, with explicit treatment of the case wherein an unperturbed resonant frequency is degenerate. A rotated Y-cut quartz resonator with degenerate thickness-shear modes (b-mode/c-mode crossover point) is analyzed as an example. Results indeed show that biasing fields can cause a degenerate frequency to split into slightly different frequencies. This phenomenon may be responsible in part for the jump discontinuities in frequency temperature curves and other frequency jump phenomena.**

I. INTRODUCTION

Piezoelectric resonators are widely used in commercial and military electronics for the purposes of frequency control and selection, and precise timing and synchronization. The most basic level of the design of piezoelectric resonators requires knowledge involving frequency analysis of ideal resonators operating at room temperature free from temperature change, stress and strain, etc., which may be called the zero-order analysis and is represented by Mindlin's early work. However, devices designed at this level are typically deficient in most applications, and knowledge of the first-order effects on frequency stability due to environmental effects like temperature change, force, and acceleration which cause biasing deformations in resonators are typically required for a successful design. The first-order effects can be predicted by Tiersten's first-order perturbation integral [1]. The perturbation integral is based on the linear theory for small fields superposed on finite biasing fields in an electroelastic body [2], which is a consequence of the fully nonlinear theory of electroelasticity [3]. French researchers also developed similar procedures systematically for analyzing frequency shifts in resonators [4-7].

The first-order perturbation integral [1] for resonator frequency shifts has been used widely to study the effects of various biasing fields in BAW, SAW, and STW resonators. The first-order perturbation integral is based on the assumption that the unperturbed resonant frequency is simple, i.e., there exists only one mode corresponding to the particular frequency under consideration when the biasing fields are not present. This assumption is also implicit in our recent extension of the first-order perturbation theory to higher-order perturbations [8]. Because of this assumption, the case when the unperturbed resonant frequency is degenerate, i.e., there exist two or more linearly independent modes for the same frequency when the biasing fields are not present, cannot be treated by the first-order perturbation integral in [1]. Degenerate modes usually occur when there exist certain geometric and material symmetries in the acoustic problem. For example, the fundamental b-mode (fast shear) and the fundamental c-mode (slow shear) of a rotated Y-cut quartz plate have the same frequency for certain cuts of the crystal, and the sixth b-mode of an AT-cut quartz plate has a frequency very close to the frequency of the seventh c-mode and the difference between the frequencies may disappear under certain biasing fields [6].

The presence of biasing fields may cause a degenerate frequency to split into different frequencies [9]. This phenomenon requires a separate treatment than the perturbation procedure in [1]. Consequently, in this paper we extend the perturbation theory in [1] to the case of degenerate acoustic eigenmodes within the theory of anisotropic elasticity. A general formulation of the perturbation procedure applied to the eigenvalue problem for resonant frequencies in crystal resonators under biasing deformations is given, with explicit treatment of the case wherein an unperturbed resonant frequency is degenerate. A rotated Y-cut quartz resonator with degenerate thickness-shear modes (b-mode/c-mode crossover point) is analyzed as an example. Results indeed show that biasing fields can cause a degenerate frequency to split into slightly different frequencies. This phenomenon may be responsible in part for the jump discontinuities in frequency temperature curves and other frequency jump phenomena [10]. Sec. II of this paper gives a summary of the governing equations for small, dynamic fields superposed on finite, static biasing fields in an anisotropic elastic body. A general formulation of the perturbation procedure then is given in Sec. III. In Sec. IV, calculations are performed for a rotated Y-cut quartz resonator as an example. Some discussions and comparisons are made in Sec. V. Finally, a few conclusions are drawn in Sec. VI.

II. CRYSTAL RESONATORS UNDER BIASING FIELDS

For the vibration of a crystal resonator under biasing fields, the theory for small, incremental deformations superposed on finite biasing deformations in an electroelastic body [2] is needed and is summarized in this section. For materials like quartz with weak piezoelectric coupling, the piezoelectric contribution can be neglected and an elastic analysis is sufficient for many analyses. The Cartesian tensor notation, the summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index will be used. A superimposed dot represents the time derivative.

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Consider the following three configurations of an elastic body as shown in Fig. 1.

A. The Reference Configuration

At this state the body is undeformed. A generic point is denoted by $\mathbf{X}$ with rectangular coordinates X_K. The mass density in the reference configuration is denoted by ρ_0. The body occupies a region V with boundary surface S and unit exterior normal N.

B. The Initial Configuration

In this state the body is deformed statically and finitely. The position of the material point associated with $\mathbf{X}$ is given by $x_\alpha = x_\alpha(\mathbf{X})$. The initial displacement is given by $\mathbf{w} = \mathbf{x}-\mathbf{X}$. The initial deformations are also called the biasing deformations. They satisfy the following static equations of nonlinear elasticity

$$K^0_{K\alpha,K} = 0, \quad K^0_{K\alpha} = x_{\alpha,L} T^0_{KL}, \quad T^0_{KL} = \left.\frac{\partial \Sigma}{\partial E_{KL}}\right|_{\mathbf{E}^0},$$

$$\begin{aligned} E^0_{KL} &= \left(x_{\alpha,K} x_{\alpha,L} - \delta_{KL}\right)/2 \\ &= \left(w_{K,L} + w_{L,K} + w_{N,K} w_{N,L}\right)/2, \end{aligned} \tag{1}$$

where $\Sigma = \Sigma\,(\mathbf{E})$ is the strain energy density function and $\mathbf{E}$ the finite strain tensor.

C. The Present Configuration

To the deformed body at the initial configuration, time-dependent small deformations are applied. The final position of the material point associated with $\mathbf{X}$ is given by $y_i = y_i(\mathbf{X},t)$. The small, incremental displacement vector is denoted by $\mathbf{u} = \mathbf{y}-\mathbf{x}$. The equations of motion for the incremental fields are

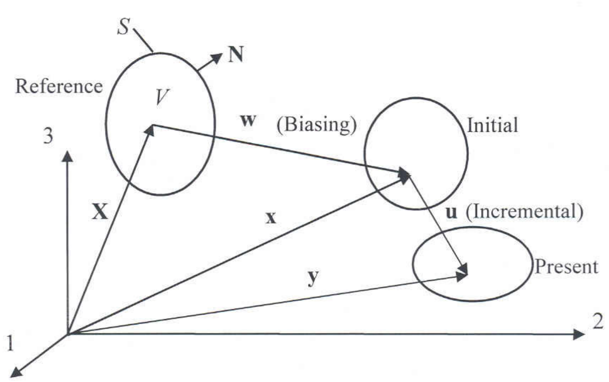

Fig. 1 The reference, initial and present configurations of an elastic body.

$$K^1_{K\alpha,K} = \rho_0 \ddot{u}_\alpha,$$

$$K^1_{K\alpha} = G_{K\alpha L\gamma} u_{\gamma,L},$$

$$G_{K\alpha L\gamma} = x_{\alpha,M} \left.\frac{\partial^2 \Sigma}{\partial E_{KM} \partial E_{LN}}\right|_{\mathbf{E}^0} x_{\gamma,N} + T^0_{KL}\delta_{\alpha\gamma} = G_{L\gamma K\alpha}, \tag{2}$$

where $\delta_{\alpha\gamma}$ is the Kronecker delta. We note that the incremental stress tensor depends linearly on the incremental displacement gradient. The effective material constants $G_{K\alpha L\gamma}$ depend on the biasing deformation $\mathbf{x}(\mathbf{X})$ or $\mathbf{w}(\mathbf{X})$ which is assumed known from solving the biasing deformation problem (1).

D. Small Biasing Deformations

In many applications the biasing deformations are infinitesimal. Small biasing deformations can be calculated from the linear theory of elasticity, or the following linear version of (1)

$$\begin{aligned} &K^0_{K\alpha,K} = 0, \quad K^0_{K\alpha} = \delta_{\alpha L} T^0_{KL}, \\ &T^0_{KL} = c_{IJKL} E^0_{KL}, \quad E^0_{KL} = \left(w_{K,L} + w_{L,K}\right)/2, \end{aligned} \tag{3}$$

where $\delta_{\alpha L}$ is the Kronecker delta when the Cartesian coordinate systems of X_L and x_α are coincident, and c_{ABCD} are the second-order fundamental elastic constants. In order to include completely the lowest-order effect of biasing deformations, the following strain energy density function is needed to calculate the effective elastic constants $G_{K\alpha L\gamma}$ in (2)

$$\Sigma\left(E_{KL}\right) = \frac{1}{2} c_{ABCD} E_{AB} E_{CD} + \frac{1}{6} c_{ABCDEF} E_{AB} E_{CD} E_{EF} \tag{4}$$

where c_{ABCDEF} are the third-order fundamental elastic constants. For small biasing deformations we write the small biasing displacement as $\varepsilon \mathbf{w}$, where ε is a small parameter. Then, up to the order of ε, we have the following expression for $G_{K\alpha L\gamma}$

$$\begin{aligned} G_{K\alpha L\gamma} &= c_{K\alpha L\gamma} + \varepsilon \hat{c}_{K\alpha L\gamma}, \\ \hat{c}_{K\alpha L\gamma} &= c_{K\alpha LN} w_{\gamma,N} + c_{KML\gamma} w_{\alpha,M} \\ &\quad + c_{K\alpha L\gamma AB} w_{A,B} + c_{KLAB} w_{A,B} \delta_{\alpha\gamma}. \end{aligned} \tag{5}$$

III. Perturbation Analysis of Degenerate Resonant Frequencies

For time-harmonic problems with frequency ω, with successive substitutions, (2) can be written as an eigenvalue problem in the following form

$$\begin{aligned}&\left[\left(c_{K\alpha L\gamma}+\varepsilon\hat{c}_{K\alpha L\gamma}\right)u_{\gamma,L}\right]_{,K}+\lambda\rho_0 u_\alpha=0 \quad in \quad V,\\&u_\alpha=0 \quad on \quad S_1,\\&N_K\left(c_{K\alpha L\gamma}+\varepsilon\hat{c}_{K\alpha L\gamma}\right)u_{\gamma,L}=0 \quad on \quad S_2,\end{aligned} \tag{6}$$

where $\lambda=\omega^2$, $S_1\cup S_2=S$, $S_1\cap S_2=0$, and we have included boundary conditions. We note that the small parameter also appears in the boundary conditions. Resonant frequencies and modes are nontrivial eigen-solutions to (6). The modes can be normalized by $<\mathbf{u};\rho_0\mathbf{u}>=1$ with respect to the following inner product of two vectors

$$<\mathbf{u};\mathbf{v}>=\int_V u_\alpha v_\alpha dV \ . \tag{7}$$

We are interested in the case of an eigenvalue μ which is N-fold degenerate (with multiplicity N) when there are no biasing deformations or ε = 0, i.e., there are N linearly independent eigenvectors $U_\alpha^{(n)}$ (n = 1,2,3,…,N) corresponding to μ such that

$$\begin{aligned}&\left(c_{K\alpha L\gamma}U_{\gamma,L}^{(n)}\right)_{,K}+\mu\rho_0 U_\alpha^{(n)}=0 \quad in \quad V,\\&U_\alpha^{(n)}=0 \quad on \quad S_1,\\&N_K c_{K\alpha L\gamma}U_{\gamma,L}^{(n)}=0 \quad on \quad S_2,\\&n=1,2,3,\cdots,N.\end{aligned} \tag{8}$$

We want to determine how μ and $U_\alpha^{(n)}$ are affected by the biasing deformations as governed by (6). In this case with degeneracy we cannot expect the individual eigenvectors to vary continuously with ε unless we have selected the system of unperturbed eigenvectors for the degenerate eigenvalue in a suitable way [9]. Accordingly, we assume [9]

$$\begin{aligned}&\lambda\cong\mu+\varepsilon\nu,\\&u_\alpha\cong\sum_{n=1}^{N}\gamma^{(n)}U_\alpha^{(n)}+\varepsilon V_\alpha,\end{aligned} \tag{9}$$

where ν and V_α are perturbations of the eigenvalue and the eigenvectors due to the biasing deformations, and $\gamma^{(n)}$ are undetermined constants. Substituting (9) into (6), collecting coefficients of terms linear in ε, we obtain

$$\begin{aligned}&(c_{K\alpha L\gamma}V_{\gamma,L})_{,K}+(\hat{c}_{K\alpha L\gamma}\sum_{n=1}^{N}\gamma^{(n)}U_{\gamma,L}^{(n)})_{,K}\\&\qquad+\mu\rho_0 V_\alpha+\nu\rho_0\sum_{n=1}^{N}\gamma^{(n)}U_\alpha^{(n)}=0 \quad in \quad V,\\&V_\alpha=0 \quad on \quad S_1,\\&N_K(c_{K\alpha L\gamma}V_{\gamma,L}+\hat{c}_{K\alpha L\gamma}\sum_{n=1}^{N}\gamma^{(n)}U_{\gamma,L}^{(n)})=0 \quad on \quad S_2.\end{aligned} \tag{10}$$

Multiplying both sides of $(10)_1$ by $U_\alpha^{(m)}$ through the inner product in (7), with integration by parts, using (8) and the boundary conditions $(10)_{2,3}$, we have

$$\sum_{n=1}^{N}\left(A^{(mn)}-\nu B^{(mn)}\right)\gamma^{(n)}=0\,, \tag{11}$$

where

$$\begin{aligned}&A^{(mn)}=\int_V \hat{c}_{K\alpha L\gamma}U_{\alpha,K}^{(m)}U_{\gamma,L}^{(n)}dV=A^{(nm)},\\&B^{(mn)}=\int_V \rho_0 U_\alpha^{(m)}U_\alpha^{(n)}dV=<\rho_0\mathbf{U}^{(m)};\mathbf{U}^{(n)}>=B^{(nm)}.\end{aligned} \tag{12}$$

If the unperturbed eigenvectors are normalized by $<\rho_0\mathbf{U}^{(m)};\mathbf{U}^{(n)}>=\delta^{(mn)}$, where $\delta^{(mn)}$ is the Kronecker delta, then $B^{(mn)}=\delta^{(mn)}$. From (11), perturbation of the eigenvalue can be determined as roots of the characteristic equation

$$|\,A^{(mn)}-\nu B^{(mn)}\,|=0\,. \tag{13}$$

The above perturbation analysis is for the eigenvalue $\lambda=\omega^2$. For ω we make the following expansion

$$\omega\cong\omega_0+\varepsilon\varpi\,, \tag{14}$$

where ω_0 is the frequency when ε = 0, or when the biasing fields are not present. Then

$$\lambda=\omega^2\cong(\omega_0+\varepsilon\varpi)^2\cong\mu+\varepsilon\nu\,. \tag{15}$$

From the terms linear in ε in (15) we have

$$\varpi=\frac{\nu}{2\omega_0}\,. \tag{16}$$

IV. AN EXAMPLE

As an example of the perturbation procedure in Sec. III, we consider thickness-shear vibrations of a rotated Y-cut quartz plate (Fig. 2) under biasing deformations.

A. Biasing Deformations

The biasing deformations are assumed to be small and are governed by the linear theory of anisotropic elasticity (3). We consider biasing deformations caused by the following planar state of stress

$$\begin{aligned} &T_2^0 = T_4^0 = T_6^0 = 0, \\ &T_1^0, \quad T_3^0, \quad and \quad T_5^0 \quad are\ constants. \end{aligned} \tag{17}$$

We note that the above state of stress satisfies the traction free boundary conditions at the major surfaces of the plate, the equilibrium and the compatibility equations of elasticity and is therefore a stress solution to the equations of elasticity. The nature of this state of planar stress is not specified here, which may be due to electrode film stresses or averages of stresses due to a pair of applied diametrical forces, etc. The strain components corresponding to (17) can be calculated from

$$E_p^0 = s_{pq} T_q^0, \quad p,q = 1,2,\cdots,6\,, \tag{18}$$

where s_{pq} is the compliance matrix. For rotated Y-cut quartz values of s_{pq} can be found from [11] and tensor transformation. It has been known that the change in the resonant frequency due to a *homogeneous*, infinitesimal rigid biasing rotation vanishes. Therefore we select the homogeneous rigid rotation to vanish. Then the biasing displacement gradients are simply given by [12]

$$\varepsilon w_{K,L} = E_{KL}^0\,. \tag{19}$$

For rotated Y-cut quartz the stiffness matrix takes the following form [13]

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix}. \tag{20}$$

In such a case, instead of using the compliance matrix, the stiffness matrix can be easily inverted to obtain expressions for strains in terms of stresses

$$\begin{aligned} &E_1^0 = \frac{\lambda_4 T_1^0 - \lambda_3 T_3^0}{\lambda_1\lambda_4 - \lambda_2\lambda_3}, \quad E_3^0 = \frac{\lambda_2 T_1^0 - \lambda_1 T_3^0}{\lambda_2\lambda_3 - \lambda_1\lambda_4}, \\ &E_5^0 = \frac{c_{66}}{c_{55}c_{66} - c_{56}^2} T_5^0, \quad E_2^0 = -\nu_{21}E_1^0 - \nu_{23}E_3^0, \\ &E_4^0 = -\nu_{41}E_1^0 - \nu_{43}E_3^0, \quad E_6^0 = \frac{c_{56}}{c_{56}^2 - c_{55}c_{66}} T_5^0, \end{aligned} \tag{21}$$

where

$$\begin{aligned} &\lambda_1 = c_{11} - c_{12}\nu_{21} - c_{14}\nu_{41}, \quad \lambda_2 = c_{13} - c_{23}\nu_{21} - c_{34}\nu_{41}, \\ &\lambda_3 = c_{13} - c_{12}\nu_{23} - c_{14}\nu_{43}, \quad \lambda_4 = c_{33} - c_{23}\nu_{23} - c_{34}\nu_{43}, \\ &\nu_{21} = \frac{c_{12}c_{44} - c_{14}c_{24}}{c_{22}c_{44} - c_{24}^2}, \quad \nu_{23} = \frac{c_{23}c_{44} - c_{24}c_{34}}{c_{22}c_{44} - c_{24}^2}, \\ &\nu_{41} = \frac{c_{14}c_{22} - c_{12}c_{24}}{c_{22}c_{44} - c_{24}^2}, \quad \nu_{43} = \frac{c_{34}c_{22} - c_{23}c_{24}}{c_{22}c_{44} - c_{24}^2}. \end{aligned} \tag{22}$$

B. Unperturbed Thickness-Shear Modes

Consider the plate in Fig. 2, which is unbounded in the X_1 and X_2 directions. The two major surfaces of the plate are traction free. For such a plate harmonic thickness vibrations independent of X_1 and X_2 exist. The displacement equations of motion for thickness vibrations can be written in the following form corresponding to (8)

$$\begin{aligned} &c_{66}U_{1,22} + \mu\rho_0 U_1 = 0, \\ &c_{22}U_{2,22} + c_{24}U_{3,22} + \mu\rho_0 U_2 = 0, \\ &c_{24}U_{2,22} + c_{44}U_{3,22} + \mu\rho_0 U_3 = 0, \end{aligned} \tag{23}$$

where μ is the square of the unperturbed frequency. From (23) and the traction free boundary conditions the fundamental mode for thickness-shear in the X_1 direction (denoted by TSh1) is found to be

$$\mu = \frac{\pi^2 c_{66}}{4\rho_0 h^2}, \quad \mathbf{U}^{(1)} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} A \sin\frac{\pi X_2}{2h}\,. \tag{24}$$

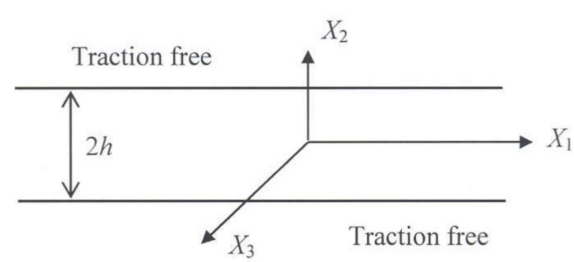

Fig. 2 A rotated Y-cut quartz plate.

We consider a portion of the plate of length $2a$ and width $2b$ in the X_1 and X_2 directions, respectively. Then the coefficient A in (24) can be determined from the normalization condition $<\rho_0 \mathbf{U}^{(1)};\mathbf{U}^{(1)}>=1$ as

$$A=\frac{1}{\sqrt{4\rho_0 abh}}\,. \tag{25}$$

It can be seen from below that the above choice of a and b for normalization will not affect the solutions on frequency shifts. For rotated Y-cut quartz, the thickness-shear in the X_3 direction is coupled to flexure. In addition to u_3, there is also a u_2 displacement component which is usually small. It may be called a quasi-thickness-shear mode and may be denoted by TSh3. The fundamental quasi-thickness-shear in the X_3 direction is found to be

$$\mu=\frac{\pi^2\bar{c}_{44}}{4\rho_0 h^2},\quad \mathbf{U}^{(2)}=\begin{Bmatrix}0\\ \xi\\ 1\end{Bmatrix}B\sin\frac{\pi X_2}{2h}, \tag{26}$$

where

$$\begin{aligned}
&\bar{c}_{44}=\frac{1}{2}\left[c_{44}+c_{22}-\sqrt{(c_{22}-c_{44})^2+4c_{24}^2}\right],\\
&B=\frac{1}{\sqrt{1+\xi^2}}A,\quad \xi=\frac{c_{24}}{\bar{c}_{44}-c_{22}},
\end{aligned} \tag{27}$$

and ξ is usually a small number. $\bar{c}_{44}$ may be called an effective shear elastic constant. c_{66} and $\bar{c}_{44}$ or the two frequencies in (24) and (26) are different in general and they depend on θ of the rotated-Y-cut plate as shown in Fig. 3. It can be seen that when $\theta = 23.7^{\circ}$ we have $c_{66}=\bar{c}_{44}$ and the two frequencies become equal. In this case the modes in (24) and (26) are modes of the same degenerate frequency.

C. Frequency Shifts

From (12), (24) and (26) we have

$$\begin{aligned}
&A^{(11)}=\mu\frac{\hat{c}_{2121}}{c_{66}},\\
&A^{(12)}=\frac{\mu}{\sqrt{1+\xi^2}}\frac{\xi\hat{c}_{2122}+\hat{c}_{2123}}{c_{66}}=A^{(21)},\\
&A^{(22)}=\frac{\mu}{1+\xi^2}\frac{\xi^2\hat{c}_{2222}+2\xi\hat{c}_{2223}+\hat{c}_{2323}}{c_{66}},\\
&B^{(mn)}=\delta^{(mn)},
\end{aligned} \tag{28}$$

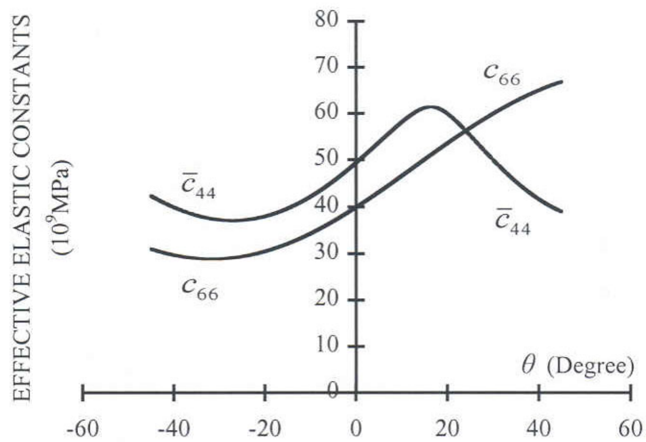

Fig. 3 c_{66} and $\bar{c}_{44}$ for a rotated Y-cut quartz plate as a function of the orientation angle θ.

where

$$\begin{aligned}
\hat{c}_{2121}&=(2c_{66}+c_{21}+c_{661})E_1^0+(c_{22}+c_{662})E_2^0\\
&\quad+(c_{23}+c_{663})E_3^0+(c_{24}+c_{664})E_4^0\\
&\qquad+(c_{46}+c_{25}+c_{665})E_5^0+(2c_{26}+c_{666})E_6^0,\\
\hat{c}_{2122}&=(c_{26}+c_{126})E_1^0+(c_{26}+c_{226})E_2^0+c_{236}E_3^0\\
&\quad+(c_{46}/2+c_{246})E_4^0+(c_{24}/2+c_{256})E_5^0\\
&\qquad+(c_{22}/2+c_{66}/2+c_{266})E_6^0,\\
\hat{c}_{2123}&=(c_{46}+c_{146})E_1^0+c_{246}E_2^0+(c_{46}+c_{346})E_3^0\\
&\quad+(c_{26}/2+c_{446})E_4^0+(c_{44}/2+c_{66}/2+c_{456})E_5^0\\
&\qquad+(c_{24}/2+c_{466})E_6^0,\\
\hat{c}_{2222}&=(c_{21}+c_{221})E_1^0+(3c_{22}+c_{222})E_2^0+(c_{23}+c_{223})E_3^0\\
&\quad+(2c_{24}+c_{224})E_4^0+(c_{25}+c_{225})E_5^0+(2c_{26}+c_{226})E_6^0,\\
\hat{c}_{2322}&=c_{241}E_1^0+(c_{24}+c_{242})E_2^0+(c_{24}+c_{234})E_3^0\\
&\quad+(c_{22}/2+c_{44}/2+c_{244})E_4^0+(c_{26}/2+c_{245})E_5^0\\
&\qquad+(c_{46}/2+c_{246})E_6^0,\\
\hat{c}_{2323}&=(c_{21}+c_{441})E_1^0+(c_{22}+c_{442})E_2^0\\
&\quad+(c_{23}+2c_{44}+c_{443})E_3^0+(2c_{24}+c_{444})E_4^0\\
&\qquad+(c_{25}+c_{46}+c_{445})E_5^0+(c_{26}+c_{446})E_6^0.
\end{aligned} \tag{29}$$

The eigenvalue problem (11) takes the following form

$$\begin{bmatrix} A^{(11)}-\nu & A^{(12)} \\ A^{(12)} & A^{(22)}-\nu \end{bmatrix}\begin{Bmatrix} \gamma^{(1)} \\ \gamma^{(2)} \end{Bmatrix}=0\,. \tag{30}$$

From (30) two eigenvalues and eigenvectors can be obtained which determine the frequency shifts and the corresponding unperturbed modes.

For quartz ρ = 2649 kg/m^3. The second-order elastic constants, in units of giga Pascals (GPa), with respect to the crystallographic axes are [13]

$$c_{pq}=\begin{pmatrix} 86.74 & 6.99 & 11.91 & -17.91 & 0 & 0 \\ 6.99 & 86.74 & 11.91 & 17.91 & 0 & 0 \\ 11.91 & 11.91 & 107.2 & 0 & 0 & 0 \\ -17.91 & 17.91 & 0 & 57.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.94 & -17.91 \\ 0 & 0 & 0 & 0 & -17.91 & 39.88 \end{pmatrix} \tag{31}$$

The fourteen independent third-order elastic constants of quartz, also in units of GPa are [14]

$$\begin{aligned} &c_{111}=-210,\quad c_{112}=-345,\quad c_{113}=+12,\quad c_{114}=-163,\\ &c_{123}=-294,\quad c_{124}=-15,\quad c_{133}=-312,\quad c_{134}=+2,\\ &c_{144}=-134,\quad c_{155}=-200,\quad c_{222}=-332,\quad c_{333}=-815,\\ &c_{344}=-110,\quad c_{444}=-276 \end{aligned} \tag{32}$$

The other seventeen nonzero third-order elastic constants are given by the following relations [15]

$$\begin{aligned} &c_{122}=c_{111}+c_{112}-c_{222}, \qquad c_{156}=\frac{1}{2}(c_{114}+3c_{124}),\\ &c_{166}=\frac{1}{4}(-2c_{111}-c_{112}+3c_{222}), \qquad c_{224}=-c_{114}-2c_{124},\\ &c_{256}=\frac{1}{2}(c_{114}-c_{124}), \qquad c_{266}=\frac{1}{4}(2c_{111}-c_{112}-c_{222}),\\ &c_{366}=\frac{1}{2}(c_{113}-c_{123}), \qquad c_{456}=\frac{1}{2}(-c_{144}+c_{155}),\\ &c_{223}=c_{113}, \quad c_{233}=c_{133}, \quad c_{234}=-c_{134},\\ &c_{244}=c_{155}, \quad c_{255}=c_{144}, \quad c_{355}=c_{344},\\ &c_{356}=c_{134}, \quad c_{455}=-c_{444}, \quad c_{466}=c_{124}. \end{aligned} \tag{33}$$

From (31-33) and the tensor transformation rules the second- and third-order elastic constants of rotated Y-cut quartz can be obtained systematically for any θ.

In Fig. 4, the frequency shift of the two modes in (24) and (26) due to T_1^0 $(T_3^0=T_5^0=0)$ is shown. As expected, the biasing stress causes the degenerate frequency of the two modes to split into two slightly different frequencies. Under the assumption of small biasing fields, with the third-order elastic constants for a complete prediction of the first-order effect of the biasing fields, the dependence of frequencies on biasing fields is linear. Frequency shifts due to a single stress component T_3^0 or T_5^0 are shown in Figs. 5 and 6, respectively. Frequency shift due to planar isotropic stress $T_1^0=T_3^0=T_0$ ($T_5^0=0$)

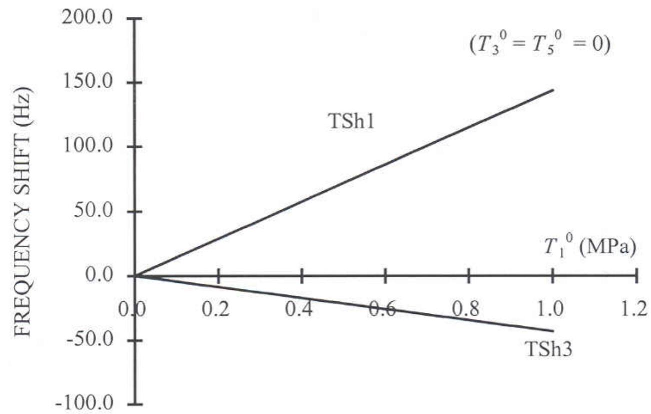

Fig. 4 Frequency shifts versus T_1^0 $(T_3^0=T_5^0=0)$.

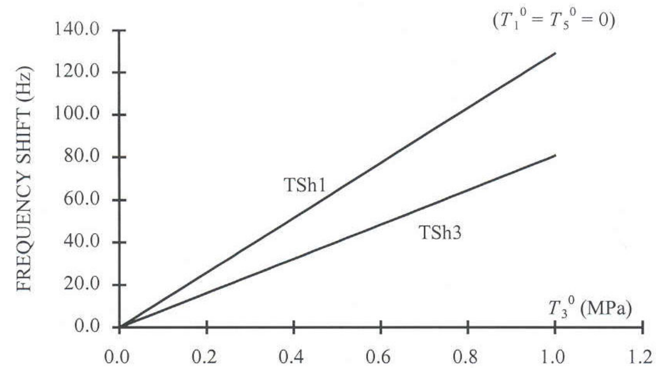

Fig. 5 Frequency shifts versus T_3^0 $(T_1^0=T_5^0=0)$.

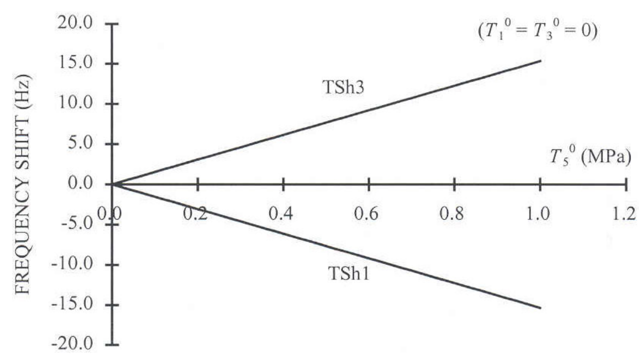

Fig. 6 Frequency shifts versus T_5^0 $(T_1^0=T_5^0=0)$.

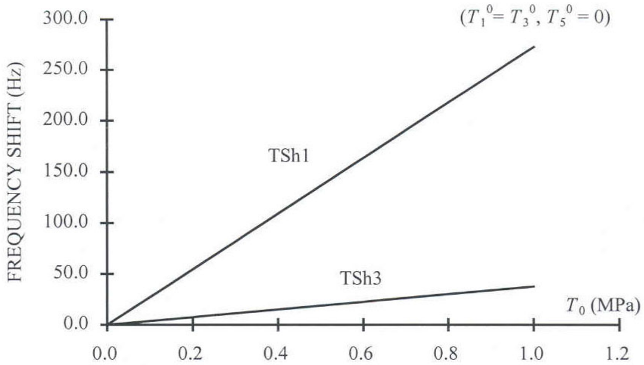

Fig. 7 Frequency shifts versus planar isotropic stress $T_1^0 = T_3^0 = T_0$ ($T_5^0 = 0$).

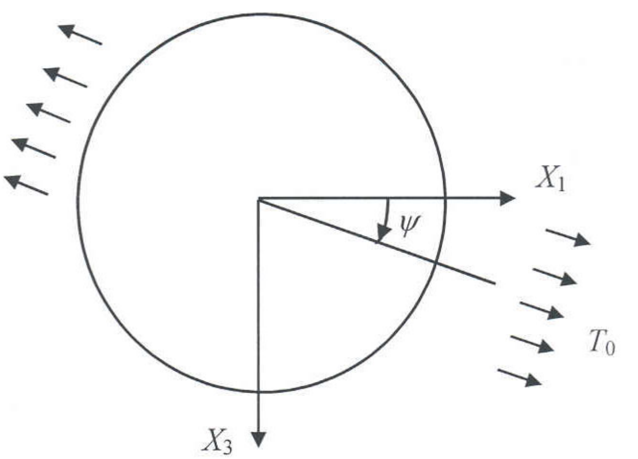

Fig. 8 Uni-axial stress T_0.

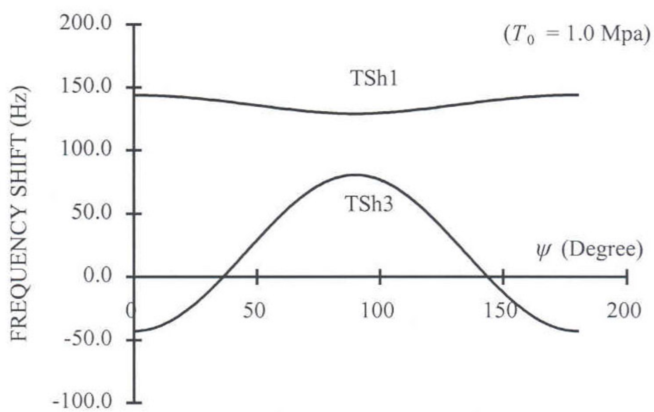

Fig. 9 Frequency shifts versus the orientation ψ of uni-axial stress T_0.

is shown in Fig. 7. We note that biasing stresses may cause the frequencies to shift in the same or opposite directions.

Fig. 8 shows a plate under uni-directional stress T_0 at an angle ψ. The tensor components of such a stress state are

$$
\begin{aligned}
T_1^0 &= \frac{T_0}{2}(1 + cos\,2\psi) \\
T_3^0 &= \frac{T_0}{2}(1 - cos\,2\psi) \\
T_5^0 &= \frac{T_0}{2} sin\,2\psi
\end{aligned}
\tag{34}
$$

The frequency shift versus ψ for a fixed value of T_0 is shown in Fig. 9. The curves are periodic with a period of π.

V. Discussion

The piezoelectric plate resonator is, in the most basic sense, comprised of a traveling wave in a propagation medium and a confinement structure. In this case, the piezoelectric substrate is the propagation medium. The confinement structure is formed by the abrupt acoustic impedance mismatch between the propagation medium and air or vacuum at the surfaces of the plate. This view is significant in that we immediately recognize similarities between acoustic and atomic resonators. For example, aside from subtle differences in the propagation medium, the infinite flat-plate acoustic resonator is fundamentally similar to the "particle-in-a-box" model of an atomic resonator.

The perturbation approaches used to analyze more complicated acoustic and atomic resonators are also similar. However, one of the key differences between the acoustic and atomic cases is the presence of symmetry-related degeneracies in atomic resonators (e.g., three equivalent p-orbitals). A corresponding phenomenon has yet to be identified in piezoelectric resonators, hence the perturbation theory for acoustic resonators was initially developed for distinct eigenvalues, while that for atomic resonators has been developed taking into account degenerate eigenvalues. In fact, some of the well known band-splitting effects in atomic resonators (e.g. Stark and Zeeman effects) arise directly from the application of an external bias to a degenerate resonance.

The degenerate acoustic eigenmodes studied in this paper are largely of academic interest. Of greater significance are experimental data on frequency jumps such as in Fig. 10 [16]. Such data appear to indicate the presence of an acoustic resonator fine structure, which in the case of Fig. 10 would be two stable states separated by $\Delta f/f \sim 10^{-10}$. In order establish a theoretical basis for such effects, we have here developed an appropriate perturbation theory for degenerate acoustic eigenmodes. In subsequent work, we will examine possible sources of previously unrecognized degenerate acoustic eigenmodes, and the resulting possibilities for an acoustic resonator fine structure.

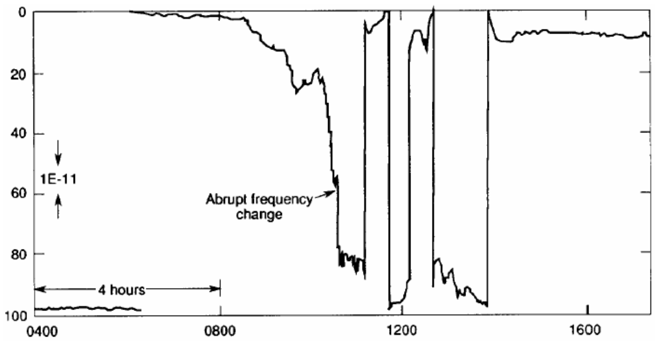

Figure 10. Abrupt frequency changes in 5-MHz BVA resonators [16].

It is worthwhile to note that the mathematics underlying the perturbation effects are generic and applicable to any of the myriad of possible biasing stresses, whether electrical, magnetic, mechanical, thermal, etc. In this sense, the perturbation approach provides an elegant means to state a "unified theory of deformation sensitivity" in which each potential bias is a special case corresponding to the specific cause of the present deformation, but the effects of a particular deformation are independent of how it was caused.

VI. Conclusions

We have obtained a general formulation for perturbation of a degenerate resonant frequency of a crystal resonator. It is shown by an example that a degenerate frequency may split into different frequencies due to biasing fields. The formulation in this paper for a degenerate frequency in an anisotropic elastic body is being generalized to electroelastic bodies by the same authors.

Acknowledgement

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